Pinnings of algebraic groups

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Contents

1 Motivation - classification of reductive algebraic groups

First we'll just talk about some motivation for discussing pinnings, which is the classification of reductive algebraic groups. Fix a field k.

Definition 1.1. An **algebraic k-group** is a group $G = G(k)$ which also an algebraic variety, such that multiplication and inversion are regular maps. (This is a more classical viewpoint, where we conflate a group scheme with its group of k-points.) As it is a variety, it is defined by polynomial equations with coefficients in k, so we can also speak of $G(L)$ for any field extension L/k, where $G(L)$ is the group of solutions in L.

Example 1.2. $\mathbb{G}_a(k), \mathbb{G}_m(k), \mathrm{GL}_n(k), \mathrm{SL}_n(k)$ are algebraic k-groups.

Definition 1.3. Let $\overline{k}$ be an algebraic closure. An **algebraic torus** is an algebraic k-group $T(k)$ such that $T(\overline{k})$ is isomorphic to a product of copies of $\mathbb{G}_m(\overline{k})$. $T(k)$ is a **split torus** if $T(k)$ is isomorphic to a product of copies of $\mathbb{G}_m(k)$.

Example 1.4. The diagonal subgroup of $\mathrm{GL}_n(k)$ or of $\mathrm{SL}_n(k)$ is a split torus.

Definition 1.5. Let U be an algebraic k-group. U is **unipotent** if it is isomorphic to a subgroup of upper triangular matrices in $\mathrm{GL}_n(k)$ with 1's along the diagonal.

Definition 1.6. Let G be an algebraic k-group. G is **reductive** if every smooth connected unipotent normal subgroup of G is trivial. This hypothesis won't be directly relevant for anything in this talk, so you can just forget about it and think of "reductive" as a synonym for "nice."

Example 1.7. $\mathbb{G}_m(k), \mathrm{SL}_n(k), \mathrm{GL}_n(k)$ are all reductive. $\mathbb{G}_a(k)$ is not reductive, since it is unipotent.

Definition 1.8. Let G be a reductive k-group. From general theory, we know that G contains a torus. Let $T \subset G$ be a maximal torus (with respect to inclusion). G is **split** if T is a split torus.

Theorem 1.9 (Classification of split reductive groups)**.** *Let k be a field. Split reductive k-groups are determined by their root data, in the sense that there is an equivalence of categories*

$$\{\text{split reductive } k\text{-groups}\} \simeq \{\text{root data}\}$$

Remark 1.10. The theorem extends to non-split reductive groups, in the sense that every non-split group is a twisted form of a split group. So a non-split group is determined by its associated split group, along with some group cohomology data.

Remark 1.11. The correspodence is roughly as follows. Take a group G with split maximal torus T. G has an associated Lie algebra $\mathfrak{g}$, the tangent space at the identity. This Lie algebra has an associated root system Φ. This Φ is the primary piece of information for a root datum. The root data also includes the character group $X^*(T) = \mathrm{Hom}(T, \mathbb{G}_m)$, cocharacter group $X_*(T) = \mathrm{Hom}(\mathbb{G}_m, T)$, and a bilinear pairing $X^*(T) \times X_*(T) \to \mathbb{Z}$.

Definition 1.12. Let G be a split reductive k-group, with split maximal torus T and associated root system is $\Phi \subset X^*(T) = \mathrm{Hom}(T, \mathbb{G}_m)$. A **pinning** of G consists of, for each $\alpha \in \Phi$, an embedding

$$X_\alpha : \mathbb{G}_a(k) \to G$$

called a **pinning map**, such that for all $t \in T$,

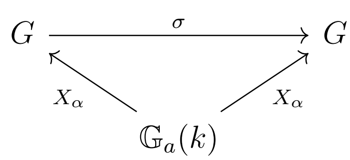
$$t \cdot X_\alpha(v) \cdot t^{-1} = X_\alpha(\alpha(t)v)$$

and that G is generated (as a group) by T along with the images of all of the X_α and T.

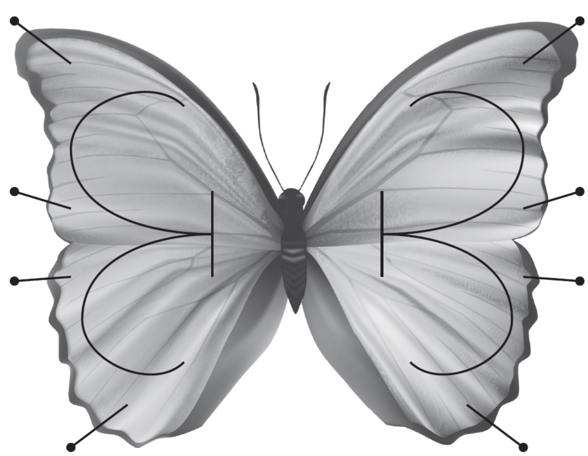
What does this mean? Well since $\alpha \in \Phi = \mathrm{Hom}(T, \mathbb{G}_m)$, α is a group homomorphism $T \to k^\times$. So this equation is saying that when we conjugate the image of X_α by an element of the torus T, we land back in the image of X_α, and furthermore all that changed is that we multiply our input $v \in k$ by $\alpha(t) \in k^\times$.

Why are pinnings important? As Milne says, "the pinning rigidifies the group."

Theorem 1.13. *Let G be a split reductive k-group with a pinning. The only automorphism of G which respects the pinning is the identity map. More precisely, if σ is an automorphism of G such that the following diagram commutes for all α, then $\sigma = \mathrm{Id}$.*



According to Milne, Grothendieck used the following analogy to talk about the structure theory of algebraic groups. An algebraic group is like a butterfly. The body of the butterfly is the maximal torus. The wings are two opposite Borel subgroups. And the pins are the pinning maps.



2 Pinning of GL_2

Let $G = \mathrm{GL}_2(k)$.

$$\mathrm{GL}_2(k) = \{X \in \mathrm{M}_2(k) : \det X \neq 0\}$$

Let $T \subset \mathrm{GL}_2(k)$ be the diagonal subgroup.

$$T = \left\{ \begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix} : t_i \in k^\times \right\}$$

Let $\chi_i : T \to k^\times$ be the map which picks off the ith diagonal entry. So

$$\chi_i(\mathrm{diag}(t_1, t_2)) = t_i$$

The character group $X^*(T) = \mathrm{Hom}(T, k^\times)$ is free abelian of rank 2, with basis χ_1, χ_2. Let

$$\alpha_{12} = \chi_1 - \chi_2 \qquad \alpha_{12}(\mathrm{diag}(t_1, t_2)) = t_1 t_2^{-1}$$

and let $\alpha_{21} = \chi_2 - \chi_1 = -\alpha_{12}$. The root system Φ associated to $\mathrm{GL}_2(k)$ is

$$\Phi = \{\pm\alpha_{12}\} = \{\alpha_{12}, \alpha_{21}\}$$

For those in the know, this is the root system of type A_1. Now we have the necessary setup to talk about pinning maps. They are

$$X_{\alpha_{12}} : \mathbb{G}_a(k) \to \mathrm{GL}_2(k) \qquad u \mapsto \begin{pmatrix} 1 & u \\ & 1 \end{pmatrix}$$
$$X_{\alpha_{21}} : \mathbb{G}_a(k) \to \mathrm{GL}_2(k) \qquad u \mapsto \begin{pmatrix} 1 & \\ u & 1 \end{pmatrix}$$

These maps have several important properties. First, they are injective group homomorphisms. That is,

$$X_{\alpha_{12}}(u + v) = X_{\alpha_{12}}(u) \cdot X_{\alpha_{12}}(v)$$

and similarly for $X_{\alpha_{21}}$. Secondly and more importantly, they interact in a favorable way with the conjugation action of the torus T. We can let $\mathrm{GL}_2(k)$ act on itself by conjugation, and then just restrict the action to $T(k)$, so we're thinking of $T(k)$ acting on $\mathrm{GL}_2(k)$ by conjugation.

$$T \times \mathrm{GL}_2(k) \to \mathrm{GL}_2(k) \qquad t \cdot X = tXt^{-1}$$

What happens when $T(k)$ acts on U_α, the image of X_α? Something very nice. Let $t = \mathrm{diag}(t_1, t_2)$ and act on $X_{\alpha_{12}}(u)$.

$$\begin{aligned} tX_{\alpha_{12}}(u)t^{-1} &= \begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix} \begin{pmatrix} 1 & u \\ & 1 \end{pmatrix} \begin{pmatrix} t_1^{-1} & \\ & t_2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & t_1 t_2^{-1} u \\ & 1 \end{pmatrix} \\ &= X_{\alpha_{12}}\left(t_1 t_2^{-1} u\right) \\ &= X_{\alpha_{12}}(\alpha_{12}(t)u) \end{aligned}$$

The same sort of thing happens with $X_{\alpha_{21}}$.

$$tX_{\alpha_{21}}(u)t^{-1} = X_{\alpha_{21}}\left(\alpha_{21}(t)u\right)$$

Note also that $\mathrm{GL}_2(k)$ is generated by the images of $X_{\alpha_{12}}, X_{\alpha_{21}}$, along with the diagonal torus T. So we have described a pinning of $\mathrm{GL}_2(k)$.

Remark 2.1. Let's return to the butterfly diagram. The body is the diagonal torus T. The wings are the upper and lower triangular subgroups, respectively. And we have only the two pinning maps, $X_{\alpha_{12}}$ and $X_{\alpha_{21}}$.

3 Pinning of GL_n

Now let's generalize to $G = \mathrm{GL}_n$. Again, let $T \subset G$ be the diagonal subgroup, and let $\chi_i : T \to k^\times$ be the character which picks off the ith entry. Now the character group $X^*(T) = \mathrm{Hom}(T, k^\times)$ is free abelian of rank n, with basis $\chi_1, \dots, \chi_n$. For $i \neq j$, define

$$\alpha_{ij} = \chi_i - \chi_j : T \to k^\times \qquad \alpha_{ij}(t) = \chi_i(t)\chi_j(t)^{-1} = t_i t_j^{-1}$$

where $t = \mathrm{diag}(t_1, \dots, t_n) \in T$. Now the associated root system is type A_{n-1}.

$$\Phi = \{\alpha_{ij} : 1 \leq i, j \leq n, i \neq j\} \subset \mathrm{Hom}(T, k^\times)$$

And now we can define our pinning maps. For $\alpha \in \Phi$, we have maps $X_\alpha = X_{\alpha_{ij}} : k \to \mathrm{SL}_3(k)$, which take $u \in k$ to the matrix with 1's along the diagonal, u in the ijth position, and zeroes elsewhere. This matrix is sometimes denoted $e_{ij}(u)$ or e_{ij}^u. For example,

$$X_{\alpha_{13}} : k \to \mathrm{SL}_n(k) \qquad u \mapsto e_{ij}(u) \begin{pmatrix} 1 & & u & & \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}$$

These maps $X_{\alpha_{ij}}$ have the same two properties as in the previous example.

1. $X_{\alpha_{ij}}$ is an injective group homomorphism.

$$X_{\alpha_{ij}}(u + v) = X_{\alpha_{ij}}(u) \cdot X_{\alpha_{ij}}(v)$$

2. The diagonal torus T acts by conjugation on $X_{\alpha_{ij}}(u)$ by inserting $\alpha_{ij}(t)$. That is, if $t = \mathrm{diag}(t_1, \dots, t_n)$, then

$$tX_{\alpha_{ij}}(u)t^{-1} = X_{\alpha_{ij}}\big(\alpha_{ij}(t)u\big)$$

and $\mathrm{GL}_n(k)$ is generated by all of the images $X_{\alpha_{ij}}$ along with T.

3.1 Chevalley commutator formula

Now we can state third property which was not so visible in the case $n = 2$.

3. Let $\alpha, \beta \in \Phi$ such that $\beta \neq \pm\alpha$ and let $u, v \in k$. Then

$$\Big[X_\alpha(u), X_\beta(v)\Big] = \prod_{\substack{i,j>0 \\ i\alpha+j\beta\in\Phi}} X_{i\alpha+j\beta}(N_{ij}^{\alpha\beta}(u,v))$$

   for some polynomial maps $N_{ij}^{\alpha\beta} : k^2 \to k$. [1]

This is called the Chevalley commutator formula. Now for some observations about it.

(i) The formula only works when $\beta \neq \pm\alpha$, so when $n = 2$, the relation is vacuous, since the only roots are $\pm\alpha_{12}$.

(ii) Since the product is over positive integral linear combinations of α, β which lie in Φ, the formula tells us that if there are no such linear combinations of α, β in Φ, then product is trivial, so the commutator is trivial, or in other words, they commute. So for example, if we take $n = 4$ and $\alpha_{12} = \chi_1 - \chi_2$ and $\alpha_{34} = \chi_3 - \chi_4$, there is no α_{ij} which is a positive integral combination of these. So

$$\Big[X_{\alpha_{12}}(u), X_{\alpha_{34}}(v)\Big] = 1$$

Consequently, if we want to see an interesting example, a nontrivial commutator, we'll need to go to at least $\mathrm{GL}_3(k)$ and choose some roots whose sum is also a root. So let's work out such an example. Fix $n = 3$. Choose α_{12} and α_{23}. Then

$$\alpha_{12} + \alpha_{23} = \chi_1 - \chi_2 + \chi_2 - \chi_3 = \chi_1 - \chi_3 = \alpha_{13}$$

This is the only positive integral linear combination of α_{12}, α_{23} which is a root. So the commutator formula says that

$$\Big[X_{\alpha_{12}}(u), X_{\alpha_{23}}(v)\Big] = X_{\alpha_{12}}(N(u,v))$$

for some polynomial function $N : k^2 \to k$. If we do the big matrix calculation on the LHS, we get

$$\Big[X_{\alpha_{12}}(u), X_{\alpha_{23}}(v)\Big] = \begin{pmatrix} 1 & & uv \\ & 1 & \\ & & 1 \end{pmatrix} = X_{\alpha_{13}}(uv)$$

So $N(u,v) = uv$.

Remark 3.1. The Chevalley commutator formula looks intimidating and is not very succinct or memorable. The way to think about it just that it says that the additive structure of the root system Φ controls the multiplication structure of the group G.

[1] In particular, $N_{ij}^{\alpha\beta}$ is homogeneous of degree i in the first variable and homogeneous of degree j in the second variable.

4 Pinning of Sp_4

Let H be the following matrix.

$$H = \begin{pmatrix} & -1 & & \\ 1 & & & \\ & & & -1 \\ & & 1 & \end{pmatrix} \in \mathrm{GL}_4(k)$$

Now define the group

$$\mathrm{Sp}_4(k) = \left\{ X \in \mathrm{SL}_4(k) : X^t H X = H \right\}$$

Again we let $T \subset \mathrm{Sp}_4(k)$ be the subgroup of diagonal matrices. By a tedious matrix calculation, you can work out that T is not the full diagonal subgroup of $\mathrm{SL}_4(k)$. Elements of T have the form

$$\begin{pmatrix} t_1 & & & \\ & t_1^{-1} & & \\ & & t_3 & \\ & & & t_3^{-1} \end{pmatrix}$$

Again, let $\chi_i : T \to k^\times$ be the character which picks off the ith diagonal entry. Then clearly $\chi_2 = -\chi_1$ and $\chi_4 = -\chi_3$, so $\mathrm{Hom}(T, \mathbb{G}_m)$ is free abelian of rank 2, with basis χ_1, χ_3. Now our root system is

$$\Phi = \{\pm\chi_i \pm \chi_j : i, j \in \{1, 3\}\} = \{\pm 2\chi_1, \pm 2\chi_3, \pm\chi_1 \pm \chi_3\}$$

which is the root system of type C_2. And our pinning maps are

$$X_{2\chi_1}(v) = \begin{pmatrix} 1 & v & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \qquad X_{-2\chi_1}(v) = \begin{pmatrix} 1 & & & \\ v & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

$$X_{2\chi_3}(v) = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & v \\ & & & 1 \end{pmatrix} \qquad X_{-2\chi_3}(v) = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & v & 1 \end{pmatrix}$$

$$X_{\chi_1-\chi_3}(v) = \begin{pmatrix} 1 & & v & \\ & 1 & & \\ & & 1 & \\ & -v & & 1 \end{pmatrix} \qquad X_{\chi_3-\chi_1}(v) = \begin{pmatrix} 1 & & & \\ & 1 & & -v \\ v & & 1 & \\ & & & 1 \end{pmatrix}$$

$$X_{\chi_1+\chi_3}(v) = \begin{pmatrix} 1 & & & v \\ & 1 & & \\ & v & 1 & \\ & & & 1 \end{pmatrix} \qquad X_{-\chi_1-\chi_3}(v) = \begin{pmatrix} 1 & & & \\ & 1 & v & \\ & & 1 & \\ v & & & 1 \end{pmatrix}$$

These have the same three properties.

1. Each X_α is a group homomorphism.

$$X_\alpha(u+v) = X_\alpha(u)X_\alpha(v)$$

2. The diagonal subgroup $T \subset \mathrm{Sp}_4(k)$ acts by conjugation on $\operatorname{im} X_\alpha$ by inserting $\alpha(t)$.

$$tX_\alpha(v)t^{-1} = X_\alpha\Big(\alpha(t)v\Big)$$

3. (Chevalley commutator formula) For $\alpha, \beta \in \Phi$ such that $\beta \neq \pm\alpha$ and any $u, v \in k$,

$$\Big[X_\alpha(u), X_\beta(v)\Big] = \prod_{\substack{i,j>0 \\ i\alpha+j\beta\in\Phi}} X_{i\alpha+j\beta}\Big(N_{ij}^{\alpha\beta}(u,v)\Big)$$

   where $N_{ij}^{\alpha\beta} : k^2 \to k$ is a polynomial map.

5 Generalization by Petrov and Stavrova

Nothing I've said about pinnings is that new, it's all theory going back about 50 or 60 years. But in recent years (2009) in work of Petrov and Stavrova, they develop a large generalization of pinning theory. In their theorem,

1. k can be replaced by any commutative ring without idempotents.

2. The torus T is replaced by the more general notion of a Levi subgroup L_P.

3. The previous two generalizations require a more complicated construction of the relative root system Φ, which can now be a nonreduced root system or not even a classical root system at all.

4. The generalization of the equality

$$X_\alpha(u+v) = X_\alpha(u)X_\alpha(v)$$

   has an extra factor on the RHS to account for when Φ is nonreduced.

5. The generalization of the equality

$$tX_\alpha(v)t^{-1} = X_\alpha\Big(\alpha(t)v\Big)$$

   also an extra factor on the RHS to account for when Φ is nonreduced. Also, $\alpha(t)$ may instead be replaced by a linear map instead of just a scalar if t is in the Levi subgroup but not inside a torus.

6. Perhaps surprisingly, the Chevalley commutator formula doesn't really change at all in the generalized version.